

A Graphical Representation of the Multi-Domain Signal Processing

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Abstract: A new framework on the representation of signal processing that considers multi-domain information flow of the signal is proposed. Elementary information processing and their diagrams are identified to analyze the components of an algorithm, which is to be translated to the diagram. Intra- and extra-domain relationships are represented as graphs that connect the corresponding domains by suitable mappings or transforms and corresponding emissions or absorption of information. Various examples: convolution, wavelet filtering, projection imaging and coding in noisy channel; are presented to show the ubiquity and to illustrate the usability of the proposed framework.

Keywords: signal flow graph, commutative diagram, multi-domain signal processing

1. INTRODUCTION

A diagram is a visualization tool describing a particular concept or idea. Physicists use the Feynman [1, 2] diagram to describe the history of a particle (Fig.1.a), by which all interaction and scattering are explained, and corresponding mathematical formulas are derived. Mathematicians use commutative diagrams (Fig.1.b), which is a diagram of objects and morphisms. By picking two objects in the commutative diagram, one can follow any path to obtain the same result by composition [1, 3].

In Signal Processing [4], a signal flow diagram (Fig.1.c) has been used to describe the transformation of signals that is performed by a pre-defined processing stages. However, this diagram cannot illustrate signal's transition from a domain to another one, which in many cases are important to follow.

(Computed Tomography), acquired projection- domain data from X-ray sensors cannot directly be presented to a user as a meaningful picture. Transforming the data to spatial domain is required. Last but not least, transmission of signals in a digital communication system requires a transform of the signals into symbols, followed by a mapping of the symbols into a finite set of waveforms representing a particular modulation scheme.

Therefore, it is realized that the combination of commutative diagram in Mathematics and the signal flow graph in Signal Processing to form a new kind of diagram will properly represent the process and will be able to provide meaningful information. Based on this diagram, any possible paths can be inferred so that new processing techniques can be developed.

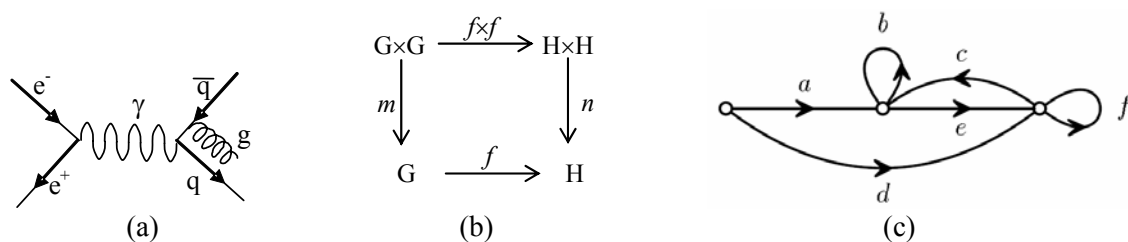


Fig.1. Various kind of diagrams found in scientific/engineering field: (a) the Feynman diagram used in Physics, (b) the commutative diagram used in Mathematics, and (c) the signal flow graph used in Signal Processing

Multidomain scheme in signal processing is required for various reasons. For example, the computational complexity of a convolution process--which is an elementary computational step in frequency-domain filtering, will be reduced when it is performed in the frequency domain, thanks to the availability of the Fast Fourier Transform (FFT) algorithm. In the projection imaging, such as in CT

The rests of the paper are organized as follows. Section II reviews some basic notations, terminologies and identify elementary information processing along with their corresponding diagrams. In Section III, we formulate the graphical representation of multidomain signal processing and present various examples to clarify the proposed concept. At last, the paper will be concluded in Section IV.

II. TRANSFORM OF INFORMATION: ELEMENTARY PROCESSES AND THEIR DIAGRAMS

2.1 Terminologies

In the Signal Processing, information is usually represented as a sequence of numbers. For our purpose, it is necessary to adopt a more generic terminology related to both of numeric and symbolic processing; i.e. the *information processing* where the *value* of information can be represented as either a sequence of numbers or a sequence of symbols. An *information processing algorithm* will manipulate and alter the value of information with the following possibilities: number-to-number, number-to-symbols symbol-to-number, or symbol-to-symbol; according to the objectives of the algorithm.

A set of values that represents a particular kind of information is defined as a *domain*. Since the values can be numeric or symbolic, the domains can be integer (**I**), real (**R**), complex (**C**), alphabets/symbols (**A**), subsets or combination (Cartesian products) of them. Information processing is basically an intra-or extra-domain mapping, which is performed under a particular set of rules.

2.2 Transform of Information

Transform is a mapping of information. In set-theory, it corresponds to *relation* and inherits its terminologies and properties. In this paper, we adopt and restate them so that standard terminologies in the signal/information processing can be elaborated. Additionally, related graphical representation in a form of diagram will be introduced.

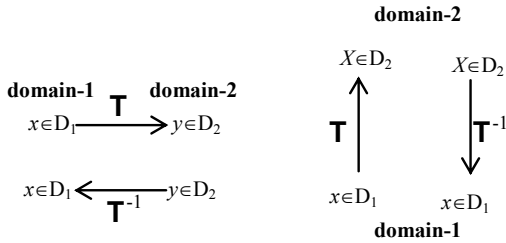


Fig.2 Diagram of a transform and its inverse

A transform maps a set of values from a domain to a co-domain. Graphically, the transform is represented by an arrow (horizontal/vertical) connecting the domain D_1 to the its co-domain D_2 . The transformation can be invertible or non-invertible ones. Fig.2 displays diagram of information transform. The transform and its inverse can be expressed as:

$$x \rightarrow y = \mathcal{A}(x) \quad (1.a)$$

$$y \rightarrow x = \mathcal{F}^1(y) \quad (1.b)$$

in the left part diagram and,

$$x \rightarrow X = \mathcal{A}(x) \quad (1.c)$$

$$X \rightarrow x = \mathcal{F}^1(y) \quad (1.d)$$

in the right part diagram.

Usually, invertibility implies both of the existence of the inverse and recoverability of original values. In Signal Processing, the invertibility corresponds only to the existence of inverse transform, while the original information may and may not fully recovered. If it is fully recovered, the transform is defined as a *lossless* transform. On the other hand, a *lossy transform* recover the original value only partially. In our diagram, the lossy transform is displayed as a transform that emits information. Figure 3 represent a multi-domain diagram of a lossy transform and its inverse that can be expressed as

$$x \rightarrow e + y = \mathcal{A}(x) \quad (2.a)$$

$$y \rightarrow x = \mathcal{F}^1(y) + e \quad (2.b)$$

in the left part diagram and,

$$x \rightarrow X = \mathcal{A}(x) + E \quad (2.c)$$

$$X \rightarrow e + x = \mathcal{F}^1(y) \quad (2.d)$$

in the right part diagram.

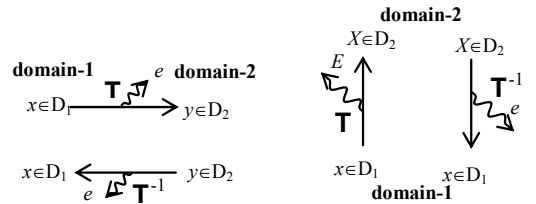


Fig.3 Diagram of lossy transform

2.3 Unitary/Orthogonal Transform

The unitary transform is a representation of signal by a set of orthogonal vectors. Included in this category are the Fourier transform, discrete cosine transform, etc. Usually, the domain of the transform represents signal values in spatio-temporal while the co-domain is its frequency. In this case, the arrow is drawn vertically to stress the difference of the domain and co-domain, which is displayed in Fig.4(a) for a Fourier transform.

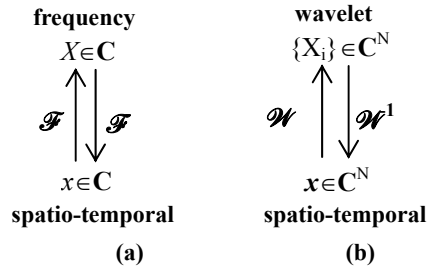


Fig.4 A diagram of (a) the Fourier transform and (b) wavelet analysis

The second example is wavelet transform. In the wavelet transform, a signal is decomposed into its multiresolution components, therefore, it is a set-to-set mapping. Fig.4(b) shows multidomain diagram of the wavelet transform.

2.4 Symbol Mapping

Symbol mapping can be considered as a transform, where a set of numerical values $r \in \mathbb{R}$ are mapped into alphabets $a \in \mathcal{A}$ and vice versa for the demapping. An example of symbol mapping is pattern recognition and the encoding-decoding process. Symbol mapping can be lossy, but the inverse is not. The corresponding diagram if symbol mapping is displayed in Fig.5, which represents a transformation of numeric-valued information to a symbolic ones.

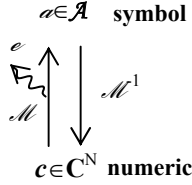


Fig.5 A diagram of symbol mapping and demapping

Symbol mapping and demapping can be expressed as mapping of numeric vector into an alphabet

$$c \rightarrow e + a = \mathcal{M}(c) \quad (3.a)$$

$$a \rightarrow c = \mathcal{M}^{-1}(a) \quad (3.b)$$

where e is mapping error.

2.5. Information Absorption

We can define an inverse of information loss, that is, *information gain* where some of missing information is recovered. Information gain is represented as information absorption. The graphical notations are displayed in Fig.6.

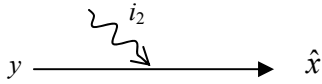


Fig.6 A diagram of information gain

2.6 Quantization

In the Digital Signal Processing (DSP), information values necessarily be represented in a finite set of numbers, typically integer ones. Therefore, real-world signal should be quantized to a set of integers and consequently information lost will occur. In our diagram, scalar quantization is represented as a signal transform from real to integer with emission of information, as displayed in Fig.7.

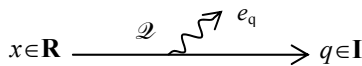


Fig.7 A diagram of scalar quantization

Quantization can be expressed as

$$x \rightarrow e_q + q = \mathcal{Q}(x) \quad (4)$$

where e_q is the quantization error.

III. GRAPHICAL REPRESENTATIONS OF MULTIDOMAIN INFORMATION PROCESSING: SOME EXAMPLES

We are now ready to represent information processing that transform values of information from a particular domain to another one by a diagram. The construction of the diagram will be started by identifying the elementary processed in the processing algorithm. Then, all domains involved in the algorithm are identified and categorized. Our approach is a simple one, that is, closely related domain will be arranged horizontally, while significantly different domain will be arranged vertically.

The *closeness* of two domains will depend on the context. For examples, numeric and symbolic domains are significantly different. The similar judgement can be made for spatial/temporal versus the transform domain. Quantization is mapping a real-valued information to integer ones, they are closed and will be arranged horizontally as mentioned before. In the following subsections, we will identify some examples of the multidomain diagram for typical signal/information processing problems.

3.1 Spatio-Temporal and Frequency Domain Convolution

Convolution is the most basic computation in the signal processing, which is used in filtering and cross-correlation algorithms. For two sequences x and h , it is performed in spatio-temporal domain by computing $f = x * h$. While in frequency domain, it is the inverse Fourier transform of $F = XH$, where X is the FT of x and H is the FT of h , accordingly. Figure 8 illustrates the convolution of $x(t)$ with $h(t)$.

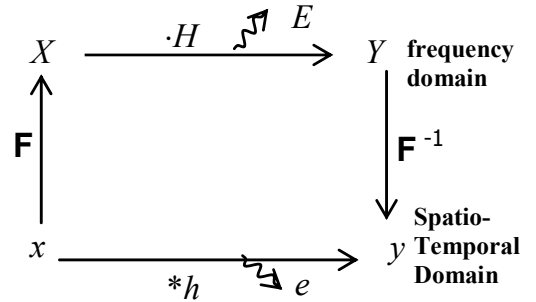


Fig.8 A graphical representation of spatio-temporal and frequency domain convolution

In the diagram, \mathcal{F} denotes the Fourier transform operator and \mathcal{F}^{-1} is its inverse. The path in the diagram can be translated into the followings. In spatio-temporal domain, it is expressed as

$$x(t) \rightarrow e(t) + y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t)h(t - \tau)d\tau \quad (5.a)$$

which is equivalent to frequency domain in the upper path

$$x(t) \rightarrow X(\omega) = \int_{-\infty}^{\infty} x(\omega) \exp(-j2\pi ft) df \quad (5.b)$$

$$X(\omega) \rightarrow E(\omega) + Y(\omega) = X(\omega) \cdot H(\omega) \quad (5.c)$$

$$Y(\omega) \rightarrow y(t) = \int_{-\infty}^{\infty} Y(\omega) \exp(j2\pi ft) df \quad (5.d)$$

where

$$H(\omega) = \int_{-\infty}^{\infty} h(\omega) \exp(-j2\pi ft) df \quad (5.e)$$

The graphical illustration is obviously illustrative, through which transform and processing formulas are derived, as shown above.

$$x(t) \leftrightarrow x_v^\mu = W\{x(t)\} = \int_{-\infty}^{\infty} x(t) \psi_v^\mu(t) dt \quad (6.a)$$

$$x_v^\mu \rightarrow e_v^\mu + y_v^\mu = \text{Threshold}\{x_v^\mu\} \quad (6.b)$$

$$y_v^\mu \rightarrow y(t) = W^{-1}\{y_v^\mu\} \\ = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_v^\mu \psi_v^\mu(t) \mu^{-2} d\mu dv \quad (6.c)$$

which is equivalent with a hypothetical lossy transform given by $f(\cdot)$, that is, $e + y = f(x)$ in the lower path.

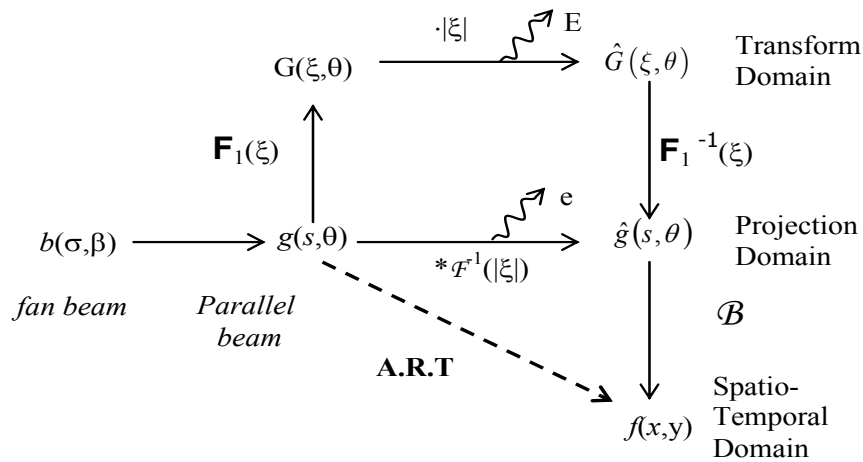


Fig.10 A diagram for signal processing in the projection imaging

3.2 Wavelet Filtering by Thresholding

In a wavelet-filtering [5], the signal is decomposed into its approximation and detail coefficient, followed by thresholding the later. The filtering result is obtained by inverse-wavelet transforming the thresholded wavelet coefficient. The thresholding obviously induce information emission and the algorithm is lossy, accordingly.

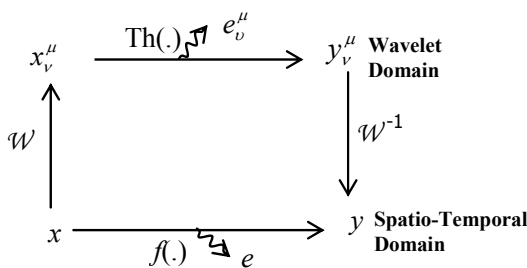


Fig.9 A diagram of wavelet-thresholding

Fig. 9 shows the multi-domain diagram, from which the following relations are derived:

3.3 Projection Imaging

A more complex diagram is shown in Fig.10, which illustrates image reconstruction from projection [6]. By following the paths, it is realized that the processing stages are expressed as follows:

The upper path:

$$b(\sigma, \beta) \rightarrow g(s, \theta) = g(R \sin \sigma, \sigma + \beta) \quad (7.a)$$

$$g(s, \theta) \rightarrow G(\xi, \theta) = F_{1,s}\{g(s, \theta)\} \\ = \int_{-\infty}^{\infty} g(s, \theta) \exp\{-j2\pi \xi s\} ds \quad (7.b)$$

$$G(\xi, \theta) \rightarrow E(\xi, \theta) + \hat{G}(\xi, \theta) = G(\xi, \theta) \cdot |\xi| \quad (7.c)$$

$$\hat{G}(\xi, \theta) \rightarrow \hat{g}(s, \theta) = F_{1,s}^{-1}\{\hat{G}(\xi, \theta)\} \\ = \int_{-\infty}^{\infty} \hat{G}(\xi, \theta) \exp\{j2\pi \xi s\} d\xi \quad (7.d.1)$$

or, hypothetical processing in the middle path that is given by:

$$g(s, \theta) \rightarrow e(s, \theta) + \hat{g}(s, \theta) = g(s, \theta) * F^{-1}\{|\xi|\} \\ = g(s, \theta) * h(s) \quad (7.d.2)$$

followed by:

$$\begin{aligned} \hat{g}(s, \theta) &\rightarrow f(x, y) = B\{\hat{g}(s, \theta)\} \\ &= \int_0^\pi \hat{g}(x \cos \theta + y \sin \theta, \theta) d\theta \end{aligned} \quad (7.e)$$

The figure also says that equivalent result can be obtained by ART (Algebraic Reconstruction Algorithm)

$$e + f(x, y) = \text{ART}(g(s, \theta)) \quad (7.f)$$

which is a direct processing in the projection domain that yields image in spatial domain.

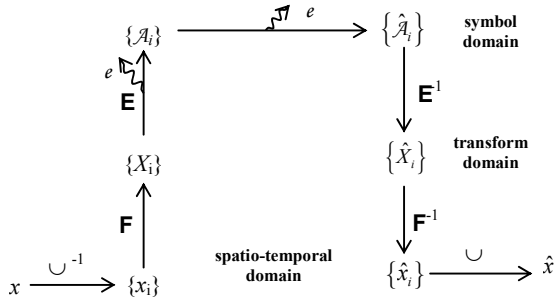


Fig.11 A diagram of coding in lossy channel

3.4 Coding of messages in a noisy channel

Figure 11 shows a multi-domain diagram of encoding-decoding process in a noisy channel. It is assumed that the signal enters the system serially. Then, the signal is groups into blocks that are represented as tiling process U^{-1} (inverse of union). The block is then transformed by an orthogonal/unitary transform (FT, DCT, etc.), followed by mapping of transform's coefficients into symbols, which is a lossy process.

In the last stage, information in a form of symbols, which is actually mapped into waveform by a digital modulation process, is transmitted into a noisy channel—which is also lossy and emits information. Decoding will take a reverse process, which is the right part of the multidomain diagram. Following the paths, we can express the coding process as:

$$x \rightarrow \{x_i\} = U^{-1}(x) \quad (8.a)$$

$$\{x_i\} \rightarrow \{X_i\} = \mathcal{A}(\{x_i\}) \quad (8.b)$$

$$\{X_i\} \rightarrow e + \{A_i\} = \mathcal{E}(\{X_i\}) \quad (8.c)$$

$$\{A_i\} \rightarrow \{\hat{A}_i\} + e \quad (8.d)$$

$$\{\hat{A}_i\} \rightarrow \{\hat{X}_i\} = \mathcal{E}^{-1}(\{\hat{A}_i\}) \quad (8.e)$$

$$\{\hat{X}_i\} \rightarrow \{\hat{x}_i\} = \mathcal{F}^{-1}(\{\hat{X}_i\}) \quad (8.f)$$

$$\{\hat{x}_i\} \rightarrow \hat{x} = U(\{\hat{x}_i\}) \quad (8.g)$$

The diagram shows that coding in a noisy channel is a lossy transform.

IV. CONCLUSIONS AND FURTHER DIRECTIONS

A new framework of signal processing considering multi-domain representation of signal flow has been presented. The multi-domain diagram, which is a combination of signal flow-graph and the commutative diagram, is capable to represent the algorithm of signal processing—or more generally—information processing, intuitively. Some examples are presented to show the applicability and ubiquity of the diagram. In the future, a more elegant and detail formulation will be explored.

ACKNOWLEDGEMENTS

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